Supersymmetric soliton in type IIB

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We will be working on supergravity theories which minimum field content is a graviton and a spin 3/2 gravitino. Therefore in addition to the Lagrangian we need to give the supersymmetry transformations

1 There are BPS naked singularities

Let us consider minimal $\mathcal{N}=2$ gauged supergravity in $D=4$ with bosonic lagrangin

$$S= \quad \mathrm{d}^4x\sqrt{-g}\left(R-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{6}{\ell^2}\right), \tag{1.1}$$

there are two gravitino $\psi^{i=1,2}$ but the KS equations can be combined into a single equation by considering $\chi=\epsilon^1+\epsilon_2$. Then KS equation is

$$D\chi+\left(\frac{1}{8}F_{\rho\sigma}\gamma^{\rho\sigma}-\frac{i}{\ell}\right)\gamma_\mu \mathrm{d}x^\mu\gamma^5\chi=0 \tag{1.2}$$

$$D\chi=\mathrm{d}\chi+\frac{1}{4}\omega_{ab}\gamma^{ab}-\frac{i}{2\ell}A\,, \tag{1.3}$$

Let us consider the a general black hole black hole with base manifold of constant curvature k

$$\mathrm{d}s^2=-f(r)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f(r)}+r^2\mathrm{d}\Sigma_k^2\,, \tag{1.4}$$

$$f(r)=\frac{r^2}{\ell^2}+k-\frac{2m}{r}+\frac{Q^2}{r^2}\,, \tag{1.5}$$

$$A=\frac{2Q}{r}\mathrm{d}t\,. \tag{1.6}$$

To study if we can solve the Killing spinor equation

$$\hat{D}\chi=(\mathrm{d}+W)\chi=0\,,$$

To ensure the solution of this equation we compute the following integrability conditions

$$\hat{D}\wedge\hat{D}\chi=\underbrace{(\mathrm{d}W+W\wedge W)}_{\Xi_{\mu\nu}}\chi=0$$

we need to ensure the existence of

$$\det \Xi_{\mu\nu} = 0\,, \tag{1.7}$$

which implies that

$$m^2 - kQ^2 = 0\,. \tag{1.8}$$

In particular for

$$\begin{aligned} k = 0 \quad & \text{plane} \quad m = 0\,, \\ k = 1 \quad & \text{spherical} \quad m = \pm Q\,, \\ k = -1 \quad & \text{hyperbolic is not BPS} \quad , \end{aligned}$$

Therefore

$$f_{k=0}(r)|_{\text{BPS}} = \frac{r^2}{\ell^2} + \frac{Q^2}{r^2} > 0\,, \tag{1.9}$$

$$f_{k=1}(r)|_{\text{BPS}} = \frac{r^2}{\ell^2} + \left(1 - \frac{Q}{r}\right)^2 > 0\,, \tag{1.10}$$

All the BPS configurations are naked singularities as the Kretchmann scalar is

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{56Q^4}{r^8} - \frac{96Q^2 m}{r^7} + \frac{48m^2}{r^6} + \frac{24}{\ell^4}\,. \tag{1.11}$$

1.1 Wick rotation of spherical electric

Let us consider $k = 1$ case in spherical coordinates

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,, \tag{1.12}$$

$$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2m}{r} + \frac{Q^2}{r^2}\,, \tag{1.13}$$

$$A = \frac{2Q}{r}\mathrm{d}t\,. \tag{1.14}$$

Considering the wick rotation

$$\begin{aligned} t &\to i\varphi \\ \phi &\to it \\ Q &\to -iQ \end{aligned}$$

then

$$\mathrm{d}s^2 = f(r)\mathrm{d}\varphi^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 - \sin^2\theta \mathrm{d}t^2) \tag{1.15}$$

$$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2m}{r} - \frac{Q^2}{r^2} \tag{1.16}$$

$$A = \frac{2Q}{r}\mathrm{d}\varphi \tag{1.17}$$

1.2 Wick rotation of spherical dyonic

Let us consider $k = 1$ case in spherical coordinates

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,, \tag{1.18}$$

$$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2m}{r} + \frac{Q^2 + P^2}{r^2}\,, \tag{1.19}$$

$$A = \frac{2Q}{r}\mathrm{d}t - 2P\cos\theta \mathrm{d}\phi\,. \tag{1.20}$$

Considering

$$\begin{aligned} t &\to i\varphi \\ \phi &\to it \\ Q &\to -iQ \\ P &\to -iP \end{aligned}$$

$$\mathrm{d}s^2 = f(r)\mathrm{d}\varphi^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 - \sin^2\theta \mathrm{d}t^2)\,, \tag{1.21}$$

$$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2m}{r} + \frac{-Q^2 - P^2}{r^2}\,, \tag{1.22}$$

$$A = \frac{2Q}{r}\mathrm{d}\varphi - 2P\cos\theta \mathrm{d}t\,. \tag{1.23}$$

The BPS conditions implies $P = Q = m = 0$ leading to

$$\mathrm{d}s^2 = \left(\frac{r^2}{\ell^2} + 1\right)\mathrm{d}\varphi^2 + \frac{\mathrm{d}r^2}{\left(\frac{r^2}{\ell^2} + 1\right)} + r^2(\mathrm{d}\theta^2 - \sin^2\theta \mathrm{d}t^2)\,, \tag{1.24}$$

$$A = 0\,. \tag{1.25}$$

1.3 Anabalon-Ross trick

There is a trick to construct regular solutions out of a planar know from long time ago ['98 Horowitz Myers]. But Anabalon and Ross prove that these regular configurations in certain situations can also be BPS.

Consider planar Schwarzschild-AdS that we consider previously with $k = -1$. Then

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}x^2 + \mathrm{d}y^2)\,, \tag{1.26}$$

$$f(r) = \frac{r^2}{\ell^2} - \frac{2m}{r} + \frac{Q^2}{r^2}\,, \tag{1.27}$$

$$A = 2Q\left(\frac{1}{r} - \frac{1}{r_0}\right)\mathrm{d}t\,. \tag{1.28}$$

To the following analytic continuation

$$t \to i\varphi\,,\; x \to it\,,\; Q \to -iQ\,, \tag{1.29}$$

which leads to

$$\mathrm{d}s^2 = -r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\overline{f}(r)} + \overline{f}(r)\mathrm{d}\varphi^2 + r^2\mathrm{d}y^2\,, \tag{1.30}$$

$$\overline{f}(r) = \frac{r^2}{\ell^2} - \frac{2m}{r} - \frac{Q^2}{r^2}\,, \tag{1.31}$$

$$A = 2Q\left(\frac{1}{r} - \frac{1}{r_0}\right)\mathrm{d}\varphi\,, \tag{1.32}$$

Note that

1. If you expand close to r_0 where $f(r_0) = 0$ you will find that for the regularity of the spacetime at $r = r_0$ you need to identify the coordinate φ as

$$\varphi \sim \varphi + \delta\,, \tag{1.33}$$

$$\delta = \frac{f'(r_0)}{4\pi} = \frac{4\pi\ell^2 r_0}{3r_0^4 + Q^2\ell^2}\,. \tag{1.34}$$

2. If we compute the integrability conditions $\det \Xi_{\mu\nu} = 0$, implies that

$$m = 0\,. \tag{1.35}$$

   note that in this case the configuration is regular! In the sense that with $m = 0$ there exist an r_0 such that $f(r_0) = 0$ since

$$f(r)|_{\mathrm{BPS}} = \frac{r^2}{\ell^2} - \frac{Q^2}{r^2}\,, \tag{1.36}$$

$$r_0 = \sqrt{\ell Q} \tag{1.37}$$

2 Type IIB and reduction

Starting from type IIB sugra in the metric-F_5 sector

$$S = \int_{\mathscr{M}_{10}} R \star 1 - \frac{1}{4} F_5 \wedge \star F_5 + \ldots (\Phi, H_3, F_1, F_3) \tag{2.1}$$

the fermions are one gravitino Ψ_μ and one dilatino λ. Setting (consistently) $\Phi = H_3 = F_1 = F_3 = 0$ we get the following SUSY variations

$$\delta\Psi_\mu \mathrm{d}x^\mu = \mathrm{d}\epsilon + \frac{1}{4}\omega_{ab}\Gamma^{ab}\epsilon + \frac{1}{16}\not{F}_5 i\sigma_2 \Gamma_a e^a \epsilon \,, \tag{2.2}$$

$$\delta\lambda = 0 \,. \tag{2.3}$$

where $\not{F}_p \equiv \frac{1}{p!} F_{a_1 \ldots a_p} \Gamma^{a_1 \ldots a_p}$. The Kaluza-Klein ansatz is ['99 Cvetic,Duff,Hoxha,Liu,Lu,L al] which will leads to a gauged $\mathcal{N} = 2$ is of the form

$$\mathscr{M}_{10} = \mathscr{M}_5 \ltimes \tilde{S}^5 \tag{2.4}$$

We use the following coordinates for the round S^5 that can be obtained by embeding a S^5 in $\mathbb{R}^6$ as

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = 1 \,, \tag{2.5}$$

the parametrzing the pairs as

$$\begin{aligned} X_1 &= \mu_1 \cos\phi_1 \,, \\ X_2 &= \mu_1 \sin\phi_1 \,, \end{aligned} \tag{2.6}$$

and so on we get the constaint

$$\sum_{i=1}^{3} \mu_i^2 = 1 \,, \tag{2.7}$$

and the induced metric is

$$\mathrm{d}s^2_{S^5_{\text{round}}} = \sum_{i=1}^{3} \mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{d}\phi_i^2 \,, \tag{2.8}$$

The five dimensional theory will contain the following fields

- 3 vectors, 2 scalars, 2 gravitino and 2 dilatino

For the reduction ansatz we will package two scalars into

$$\overrightarrow{\Phi} = (\Phi_1, \Phi_2) \tag{2.9}$$

which will be package into

$$X_i = e^{-\frac{1}{2}\overrightarrow{a}_i \cdot \overrightarrow{\Phi}} \,, \tag{2.10}$$

where

$$\vec{a}_1 = (\frac{2}{\sqrt{6}}, \sqrt{2}) \,, \quad \vec{a}_2 = (\frac{2}{\sqrt{6}}, -\sqrt{2}) \,, \quad \vec{a}_3 = (-\frac{4}{\sqrt{6}}, 0) \,. \tag{2.11}$$

The metric ansatz is

$$\mathrm{d}s^2_{10} = \tilde{\Delta}^{1/2} \mathrm{d}s^2_5 + L^2 \tilde{\Delta}^{-1/2} \sum_{i=1}^{3} X_i^{-1} \left[\mathrm{d}\mu_i^2 + \mu_i^2 (\mathrm{d}\phi_i + L^{-1} A^i)^2 \right] \,, \tag{2.12}$$

and the 5-form is

$$F_5 = G_5 + \star G_5 \,, \tag{2.13}$$

$$\begin{aligned} G_5 \equiv & \frac{2}{L} \epsilon_5 \sum_{i=1}^{3} (X_i^2 \mu_i^2 - \tilde{\Delta} X_i) - \frac{L}{2} X_i^{-1} \star_5 \mathrm{d}X_i \wedge \mathrm{d}\mu_i^2 \\ & + L^2 \sum_i X_i^{-2} \mu_i \mathrm{d}\mu_i \wedge (\mathrm{d}\phi_i + L^{-1} A^i) \wedge \star_5 F^i \,. \end{aligned} \tag{2.14}$$

2.1 Brane distribution configuration

Continuous distributions of D3-branes: Coulomb Branch ['98 Freedman, Gubser, Pilch, Warner]: Obtained by adding scalar operator of dimension $\Delta = 2$ leading to two possible manifolds manifolds $\mathscr{M}_{\pm}$ described by a dimensionless parameter $\pm\ell^2/L^2$.

$$\mathrm{d}s_{10}^2 = H^{-1/2}(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2) + H^{1/2}\sum_{i=1}^{6}\mathrm{d}y_i^6 \tag{2.15}$$

$$H(\vec{y}) = \int_{|\vec{w}|\leq\ell} \mathrm{d}^n\vec{w}\sigma_n(\vec{w})\frac{L^4}{|\vec{y}-\vec{w}|}, \quad \begin{array}{l}\sigma_2(\vec{w}) = \frac{1}{\pi\ell^2}\theta(\ell^2-\vec{w}^2)\\ \sigma_4(\vec{w}) = \frac{1}{\pi^2\ell^2}\delta^{(4)}(\ell^2-\vec{w}^2)\end{array} \tag{2.16}$$

The $n = 2$ ($\epsilon = 1$) and $n = 4$ ($\epsilon = -1$) continuous distribution has the metric

$$\begin{aligned}\mathrm{d}s_{10}^2 = &\frac{\zeta(r,\theta)r^2}{L^2}\left(\mathrm{d}x_{1,3}^2 + \frac{L^4}{r^4}\frac{\mathrm{d}r^2}{\lambda(r)^6}\right)\\ &+\frac{L^2}{\zeta(r,\theta)}(\zeta(r,\theta)^2\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\Omega_3^2 + \lambda(r)^6\sin^2\theta\mathrm{d}\Omega_1^2)\end{aligned} \tag{2.17}$$

where

$$\zeta(r,\theta) = 1 + \epsilon\frac{\ell^2}{r^2}\cos^2\theta\,, \quad \lambda(r)^6 = 1 + \epsilon\frac{\ell^2}{r^2}\,. \tag{2.18}$$

Symmetry:

$$SO(4)\times SO(2) \tag{2.19}$$

3 5D theory and the solution

The resulting theory if $D = 5$ is $\mathcal{N} = 2$ gauged sugra with Lagrangian given by

$$S = \frac{1}{2\kappa} \quad \left(R - \sum_{i=1}^{3} \frac{4}{L^2 X_i} \right) \star_5 1 - \frac{1}{2} \mathrm{d}\Phi_1 \wedge \star_5 \mathrm{d}\Phi_1 - \frac{1}{2} \mathrm{d}\Phi_2 \wedge \star \mathrm{d}\Phi_2 \,,$$
$$- \frac{1}{2} \sum_{i=1}^{3} X_i^{-2} F^i \wedge \star_5 F^i + A^1 \wedge F^2 \wedge F^3 \,. \tag{3.1}$$

This is an $\mathcal{N} = 2$ gauged sugra. The fermions are 2 dilatinos λ_1, λ_2 and 2 gravitino ψ_μ^a. The local spinorial parameter generating the SUSY transformations is a symplectic Majorana ϵ^a, $a = 1, 2$. We define a complex spinor which

$$\chi = \epsilon^1 + \mathrm{i}\epsilon^2 \,, \tag{3.2}$$

useful to integrate the Killing spinor equations. Then the Killing spinor variations for χ will be

$$(\mathrm{d} + W)\chi = 0 \,, \tag{3.3}$$
$$\Omega_1 \chi = 0 \,, \tag{3.4}$$
$$\Omega_2 \chi = 0 \,, \tag{3.5}$$

the last two are comming from the two spin $1/2$ dilatinos. Explicitly we have $\Omega_1 = \sum_i \vartheta_i \partial X_i / \partial \Phi_1$, $\Omega_1 = \sum_i \vartheta_i \partial X_i / \partial \Phi_2$

$$W = \frac{1}{4} \omega_{ab} \gamma^{ab} + L^{-1} \sum_i (-\frac{\mathrm{i}}{2} A^i + \frac{1}{3!} \gamma_c e^c X_i) \tag{3.6}$$
$$+ \frac{\mathrm{i}}{4!} (\gamma_c \gamma^{ab} - 6\delta_c^a \gamma^b) e^c \sum_i X_i^{-1} F_{ab}^i \,,$$
$$\vartheta_i = -\frac{1}{8} X_i^{-2} \gamma^{ab} F_{ab}^i - \frac{\mathrm{i}}{4} X_i^{-2} \left(\frac{\partial X_i}{\partial \Phi_1} \not{\partial} \Phi_1 + \frac{\partial X_i}{\partial \Phi_2} \not{\partial} \Phi_2 \right) + \frac{\mathrm{i}}{2L} \tag{3.7}$$

3.1 The new solution

We generalize the Anabalon-Ross solution including a non trivial VEV and non trivial currents in $D = 5$.

The seed spacetime is Particular case of static hairy electrically charged black hole. The lift of this correspond to a spinning D3 brane ['99 Cvetic,Duff,Liu,Lu,Lu,Pope,et al]. Considering coordinates $[t, x, \phi, y, z]$, the explicit configuration is given by

$$\mathrm{d}s_5^2 = \Omega(x)\left(-\mathrm{d}t^2 + \frac{\mathrm{d}x^2}{4x^2(x-1)F(x)\eta} + F(x)\mathrm{d}\phi^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{3.8}$$

$$\Omega(x) = \frac{x^{2/3}\eta}{x-1}\,, \qquad F(x) = 1 + \frac{(1-x)^2(q_1^2 - q_2^2 x)}{\eta x^2}\,.$$

The matter fields

$$\begin{aligned} &\Phi_1 = \sqrt{\frac{2}{3}}\log(x)\,, \quad \Phi_2 = 0\,, \\ &A_1 = A_2 = q_1(x^{-1} - x_0^{-1})L\mathrm{d}\phi\,, \qquad A_3 = q_2(x - x_0)L\mathrm{d}\phi \end{aligned} \tag{3.9}$$

The integration constants are (η, q_1, q_2). Let us analize the range of the coordinates $[t, x, \phi, y, z]$

1. The spacetime closed smoothly when $\exists x_0 : F(x_0) = 0$ and $\phi \in [0, \delta]$ with

$$\phi \sim \phi + \delta(q_1, q_2, \eta) \tag{3.10}$$

2. There is an asymptotic region at $x \to 1$. The sign of η determines whether $x > 1$ or $x < 1$. These two spacetimes are note diffeomorphic

$$\begin{aligned} &\eta < 0 \implies x \in]x_0, 1] \qquad \Phi_1(x) < 0 \\ &\eta > 0 \implies x \in]1, x_0] \qquad \Phi_1(x) > 0 \end{aligned} \tag{3.11}$$

3.2 Connecting with AdS

Considering the change of variable

$$x = \frac{1}{1-\eta\frac{L^2}{r^2}} \cong 1 + \eta\frac{L^2}{r^2} + o(r^{-3})\,. \tag{3.12}$$

The spacetime is asymptotically of the form

$$\mathrm{d}s_5^2 \to \frac{r^2}{L^2}(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}\phi^2) + \frac{\mathrm{d}r^2}{r^2}\,, \tag{3.13}$$

while the dilaton goes to infinity as follows

$$\Phi_1 \to \sqrt{\frac{2}{3}}\frac{L^2}{r^2} + \frac{L^4\eta^2}{3\sqrt{6}r^4} + o(r^{-8})\,, \tag{3.14}$$

Therefore by comparing with the general expansion of a scalar field in an asymptotically AdS spacetime in dimension $D = 5$ with excite a VEV in the dual theory of conformal dimension $\Delta = 2$ with VEV

$$\langle O\rangle = 2\Phi_0 = 2\sqrt{\frac{2}{3}}L^2\eta\,. \tag{3.15}$$

The relation between the mass of the scalar in AdS with the conformal weight Δ and the spacetime dimension of the dual theory d is

$$m^2L^2 = \left(\Delta - \frac{d}{2}\right)^2 - \frac{d^2}{4}\,, \tag{3.16}$$

For our case $d = 4, \Delta = 2$ we are preciselly saturating the BF bound.

$$m^2 \geq m_{BF}^2 = -\frac{d^2}{4L^2}\,. \tag{3.17}$$

A soliton solution is fully characterized in terms of its boundary conditions. So far we use the parameters (η, q_1, q_2) defining $x_0 = x_0(\eta, q_1, q_2)$ and $\delta = \delta(\eta, q_1, q_2)$. The following set of variable is physically sensible from the point of view of the

- The boundary values of the gauge fields. Which is convenient to parameterize the system in terms of a normalized 5-dimesnsion "Wilson lines"

$$\begin{aligned} \psi_1 \equiv \lim_{x\to 0} \frac{1}{2\pi L} \quad A^{1,2}\,, \qquad \psi_2 \equiv \lim_{x\to 0} \frac{1}{2\pi L} \quad A^3 \qquad (3.18)\\ \implies \qquad q_1 = \frac{2\pi x_0 \psi_1}{(x_0-1)\delta}\,, \qquad q_2 = \frac{2\pi \psi_2}{(1-x_0)\delta}\,, \end{aligned}$$

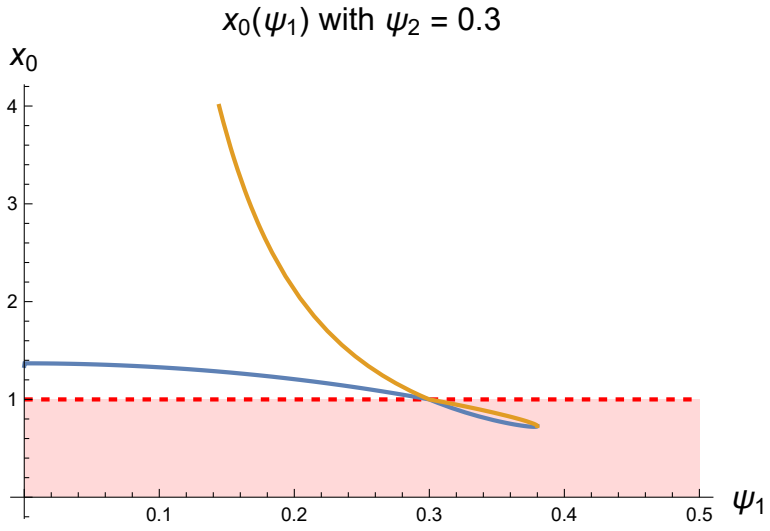
- The periodicity δ of the soliton S^1

$$\begin{aligned} &\phi \sim \phi + \delta \\ &\delta = \frac{(x_0-1)^{1/2} x_0 F'(x_0)}{2\pi} = \frac{2\pi x_0^2}{L|q_2^2 x_0^2 + q_2 x_0 - 2q_1^2|}\sqrt{\frac{\eta}{x_0-1}}\,. \end{aligned} \qquad (3.19)$$

Combing the definition of $\delta(q_1, q_2, \eta; x_0)$ with $\psi_1(x_0, \delta, q_1)$ and $\psi_2(x_0, \delta, q_2)$ we get an equation for x_0

$$\begin{aligned} \psi_1^2 x_0^3 + (\psi_2^4 + 4\psi_1^4 - 4\psi_2^2\psi_1^2 - \psi_2^2 - \psi_1^2) x_0^2 \qquad (3.20)\\ -\psi_2^2(4\psi_1^2 - 2\psi_2^2 - 1) x_0 + \psi_2^4 = 0 \end{aligned}$$

Therefore all the old variables are written in terms of (ψ_1, ψ_2, δ). There are 2 sensible solutions of x_0. With this parametrization we can cover $\eta < 0$ and $\eta > 0$.



Using the holographic renormalization procedure we can compute the energy-momentum tensor of the dual theory

$$\langle T_{ab} \rangle = -\frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{ren}}}{\delta \gamma^{ab}} \,, \tag{3.21}$$

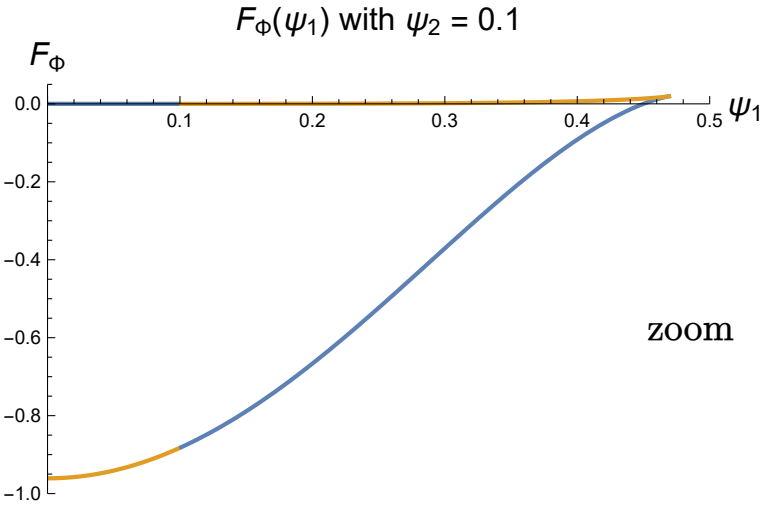
leading to

$$\begin{aligned} \langle T_{tt} \rangle &= \frac{\eta L (q_1^2 - q_2^2)}{2\kappa} \,, \\ \langle T_{yy} \rangle = \langle T_{zz} \rangle &= - \langle T_{tt} \rangle \,, \quad \langle T_{\phi\phi} \rangle = -\frac{3}{2} \langle T_{tt} \rangle \end{aligned} \tag{3.22}$$

Then we can study the free energy in the canonical ensemble is the energy $E_\Phi \equiv \langle T_{tt} \rangle V_2 \delta$

$$F_\Phi \equiv \frac{E_\Phi}{|E_0|} = -16 \frac{(\psi_1^2 x_0^2 - \psi_2^2)(\psi_1^2 x_0 - \psi_2^2)}{x_0 (x_0 - 1)^2} \tag{3.23}$$

where we are normalizing it w.r.t. the AdS soltion energy ['98 Horowitz,Myers] $E_0 = -\frac{L^3 \pi^4}{2\kappa \delta^3} V_2$



We can see that increasing ψ_1 until $|\psi_1| = |\psi_2|$ the scalar changes sing and a re-distribution of the D3-brane happens.

- Phase transition at $|\psi_1| = |\psi_2|$.
- $F_\Phi = 0$ corresponds to the energy of ['98 Freedman, Gubser, Pilch, Warner].

Let us consider a toy model to understand the different configurations we see in gravity. Consider a QFT in Minkowski 4 with $\phi \sim \phi + \delta$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}\phi^2 \tag{3.24}$$

the Casimir energy of a scalar field is

$$\langle E_{\mathrm{QFT}} \rangle = -\frac{\pi^2}{6\delta^3} V_2 X \tag{3.25}$$

X depends on the periodicity of the field in the S^1

$$X_{\mathrm{perdiodic}} = \sum_{n=1}^{\infty} (2n)^3 = \frac{8}{120}\,, \tag{3.26}$$

$$X_{\mathrm{anti-periodic}} = \sum_{n=1}^{\infty} (2n-1)^3 = -\frac{7}{120}\,.$$

If we apply this calculation to $\mathcal{N}=4$ SYM at weak coupling we find that

- For anti-periodic fermions and anti-periodic bosons we have that

$$\langle E_{\rm QFT}\rangle = -\frac{\pi^2 V_2}{6\delta^3} X_{\rm periodic}(N^2-1)(6+2-4\times 2) = 0\,, \quad (3.27)$$

- For anti-periodic fermions and periodic bosons.

$$\begin{aligned}\langle E_{\rm QFT}\rangle_{\rm mixed} &= -\frac{\pi^2 V_2}{6\delta^3} X_{\rm periodic}(N^2-1)(6+2+8\frac{7}{8}) \qquad (3.28)\\ &= -\frac{\pi^2 V_2}{6\delta^3}(N^2-1)\,,\end{aligned}$$

in the large N and considering $\frac{L^3}{\kappa} = \frac{N^2}{4\pi^2}$ we find that

$$E_0 = \frac{3}{4}\langle E_{\rm QFT}\rangle_{\rm mixed} \qquad (3.29)$$

where E_0 is the energy of the Horowitz-Myerz soliton

4 Lift to 10 dimensions

Using the lift in the deformed S^5 we show that our configuration is a solution of IIB sugra.

Change of variable

$$x = \frac{1}{1+\epsilon\frac{\ell^2}{r^2}} \qquad (4.1)$$

where $\eta = -\epsilon\ell^2/L^2$ and $\epsilon = \pm 1$. The relevant functions are

$$F(r) \equiv L^{-2} - \frac{\epsilon\ell^2 L^2}{r^4}\left(q_1^2 - q_2^2\lambda(r)^{-6}\right) \tag{4.2}$$

$$\lambda(r)^6 \equiv 1 + \epsilon\frac{\ell^2}{r^2} \tag{4.3}$$

$$\zeta(r,\theta)^2 = 1 + \epsilon\frac{\ell^2}{r^2}\cos\theta \tag{4.4}$$

The metric in 10 dimensions in coordinates $[x_{1,2}, r, \phi, \theta, \phi_3, \psi, \phi_1, \phi_2]$

$$\begin{aligned} &\mathrm{d}s_{10}^2 = \\ &\frac{\zeta(r,\theta)r^2}{L^2}\left(\frac{L^2\mathrm{d}r^2}{r^4F(r)\lambda(r)^6} + \mathrm{d}x_{1,2}^2 + F(r)L^2\mathrm{d}\phi^2\right) \\ &+ \frac{L^2}{\zeta(r,\theta)}\left\{\zeta(r,\theta)^2\mathrm{d}\theta^2 + \lambda(r)^6\sin^2\theta\left(\mathrm{d}\phi_3 + L^{-1}A_3\right)^2\right. \\ &\left.+ \cos^2\theta\left[\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\phi_1 + L^{-1}A_1\right)^2 + \cos^2\psi\left(\mathrm{d}\phi_2 + L^{-1}A_2\right)^2\right]\right\} \end{aligned} \tag{4.5}$$

while the F_5 is given by

$$F_5 = G_5 + \star G_5 \tag{4.6}$$

and $G_5 = \mathrm{d}\mathcal{C}_4$

$$\begin{aligned} \mathcal{C}_4 = &-\left[\frac{r^4}{L^4}\zeta(r,\theta)^2 + \frac{\ell^4}{r_0^2}\epsilon\cos^2\theta(q_2^2\lambda(r_0)^{-6} - q_1)\right]\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}\phi \\ &+ \ell^2\epsilon[q_1\cos^2\theta(\cos^2\psi\mathrm{d}\phi_2 + \sin^2\psi\mathrm{d}\phi_1) + \\ &\qquad + q_2\cos^2\theta\mathrm{d}\phi_3]\wedge\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z \end{aligned} \tag{4.7}$$

The flux of the F_5 on the S^5 with coord. $[\theta, \phi_3, \psi, \phi_1, \phi_2]$ with $\theta, \psi \in [0, \pi/2]$ and $\phi_1, \phi_2, \phi_3 \in [0, 2\pi]$ is

$$\int_{S^5} F_5 = \int_{S^5} \star F_5 = 4\pi^3 L^4\epsilon \tag{4.8}$$

4.1 Supersymmetry

In $D = 5$ the integrability conditions

$$\det \Omega_1 = 0\,, \qquad \det \Omega_2 = 0\,, \qquad \det(\mathrm{d}\mathbf{W} + \mathbf{W} \wedge \mathbf{W})_{\mu\nu} = 0 \tag{4.9}$$

hold iff

$$q_1 = -q_2 \tag{4.10}$$

while in $D = 10$ the $\det \Xi_{\mu\nu} = 0$ implies that

$$|q_1| = |q_2| \tag{4.11}$$

note only $q_1 = -q_2$. Regarding the KS in $D = 5$ we have that in the coordiante $x = (1 + \epsilon \ell^2/r^2)^{-1}$ when $q_1 = -q_2$ there are two complex killing spinors $\hat{\epsilon} = \epsilon^1 + \mathrm{i}\epsilon^2$

$$\Psi_1 = e^{-\frac{\mathrm{i}\pi\phi}{\delta} + \sigma(r)} \begin{pmatrix} 1 \\ \Lambda(r) \\ 0 \\ 0 \end{pmatrix}\,, \qquad \Psi_2 = e^{-\mathrm{i}\frac{\pi\phi}{\delta} + \sigma(r)} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\Lambda(r) \end{pmatrix} \tag{4.12}$$

They are anti-periodic in $\phi \in [0, \delta]$. The solution is 1/8 BPS. In this conventions AdS_5 has 4 KS spinors. In $D = 10$ was not possible construct the KS due to technical difficulties.

Summary We found AdS solitions depending on three parameters: two sources associated to gauge fields and the value of the circle S^1. In the (E, ψ_1, ψ_2) we found phase transitions between the various D3-brane distributions. The 10-dimensional uplift show that the solutions are deformations of the D3-brane distribution of the ['98 Freedman, Gubser, Pilch, Warner] leading a completely smooth configuration of type IIB. We found that the phase diagram of these solutions should correspond to the strongly coupled description of two possible vacua of the large N $\mathcal{N} = 4$ SYM.